1. Exercises from 5.3

Today we're going to practice doing different types of surface integrals. Briefly explain the concept of a surface integral.

PROBLEM 1. *(Folland 5.3.1) Find the area of the surface* $z = xy$ *inside the cylinder* $x^2 + y^2 = a^2$

- We pick a parameterization of the surface

$$g : \mathbb{R}^2 \to \mathbb{R}^3$$

$$(x, y) \mapsto (x, y, xy)$$

- Finding the infinitessimal area element:

$$dA = \left| \frac{\partial g}{\partial x} \times \frac{\partial g}{\partial y} \right| dx\, dy = \left| \begin{array}{ccc} i & j & k \\ 1 & 0 & y \\ 0 & 1 & x \end{array} \right| dx\, dy = \sqrt{1 + x^2 + y^2}\, dx\, dy$$

- The area is given by:

$$\begin{aligned} A &= \int\int_S dA \\ &= \int\int_S \left| \frac{\partial g}{\partial x} \times \frac{\partial y}{\partial \theta} \right| dx\, dy \\ &= \int\int_S \sqrt{1 + x^2 + y^2}\, dx\, dy \\ &= \int_0^{2\pi} \int_0^a r\sqrt{1 + r^2}\, dr\, d\theta \\ &= \pi \int_0^{a^2} \sqrt{1 + u}\, du \\ &= \frac{2\pi}{3} \left[(1 + a^2)^{3/2} - 1 \right] \end{aligned}$$

PROBLEM 2. *(Folland 5.3.3) Let* $0 < a < b$*, find the area of the torus obtained by revolving the circle* $(x - b)^2 + z^2 = a^2$ *around the* z*-axis.*

We are given a parameterization of the torus:

$$g : [0, 2\pi] \times [0, 2\pi] \to \mathbb{R}^3$$

$$(\theta, \phi) \mapsto ((b + a\cos\phi)\cos\theta, (b + a\cos\phi)\sin\theta, a\sin\phi)$$

So we carry through the usual construction:

$$dA = \left| \frac{\partial g}{\partial \theta} \times \frac{\partial g}{\partial \phi} \right| d\theta\, d\phi = \left| \begin{array}{ccc} i & j & k \\ -(b + a\cos\phi)\sin\theta & (b + a\cos\phi)\cos\theta & 0 \\ -a\sin\phi\cos\theta & -a\sin\phi\sin\theta & a\cos\phi \end{array} \right| d\theta\, d\phi = a(b + a\cos\phi)\, d\theta\, d\phi$$

Now we can compute the area:

$$\begin{aligned} A &= \int\int_S dA \\ &= \int\int_S \left| \frac{\partial g}{\partial \theta} \times \frac{\partial g}{\partial \phi} \right| d\theta\, d\phi \\ &= \int_0^{2\pi} \int_0^{2\pi} a(b + a\cos\phi)\, d\theta\, d\phi \\ &= 4\pi^2 ab \end{aligned}$$

PROBLEM 3. *(Folland 5.3.8(a,b,d))*

Part a): Let S be the surface $z = xy$ with $0 \le x \le 1$, $0 \le y \le 2$, oriented with the normal pointing upwards. Integrate the vector field $\mathbf{F}(x, y, z) = (xz, 0, -xy)$ over this surface.

- We first need the surface normal. The orientation is pointing UPWARDS in the $+z$-direction, so the normal is given by:

$$\mathbf{n} = \frac{\partial g}{\partial x} \times \frac{\partial g}{\partial y} = \begin{vmatrix} i & j & k \\ 1 & 0 & y \\ 0 & 1 & x \end{vmatrix} = (-y, -x, 1)$$

- We can now compute the surface integral:

$$\begin{aligned} \int\int_S \mathbf{F} \cdot n \, dA &= \int_0^1 \int_0^2 \mathbf{F}(g(x,y)) \cdot \left(\frac{\partial g}{\partial x} \times \frac{\partial g}{\partial y} \right) dy \, dx \\ &= \int_0^1 \int_0^2 \begin{pmatrix} x(xy) \\ 0 \\ -xy \end{pmatrix} \cdot \begin{pmatrix} -y \\ -x \\ 1 \end{pmatrix} dy \, dx \\ &= \int_0^1 \int_0^2 -x^2y^2 - xy \, dy \, dx \\ &= -\frac{8}{9} - 1 = -\frac{17}{9} \end{aligned}$$

Part b): $\mathbf{F}(x, y, z) = x^2\mathbf{i} + z\mathbf{j} - y\mathbf{k}$ on the unit sphere, oriented so that the normal points outward.

- For fun and practice, we'll do this problem in spherical polar coordinates. A parameterization of the sphere is:

$$g(\theta, \phi) = (\cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta), \quad \theta \in [0, \phi], \phi \in [0, 2\pi]$$

- In this coordinate system, the normal vector can be computed explicitly (intuitively, we expect it to just be the unit vector $\hat{\mathbf{r}}$):

$$\begin{aligned} \mathbf{n} &= \left| \frac{\partial g}{\partial \theta} \times \frac{\partial g}{\partial \phi} \right| \\ &= \begin{vmatrix} i & j & k \\ \cos\phi\cos\theta & \sin\phi\cos\theta & -\sin\theta \\ -\sin\phi\sin\theta & \cos\phi\sin\theta & 0 \end{vmatrix} \\ &= \sin^2\theta\cos\phi\,\mathbf{i} + \sin^2\theta\sin\phi\,\mathbf{j} + (\cos^2\phi\sin\theta\cos\theta + \sin^2\phi\sin\theta\cos\theta)\,\mathbf{k} \\ &= \sin^2\theta\cos\phi\,\mathbf{i} + \sin^2\theta\sin\phi\,\mathbf{j} + \sin\theta\cos\theta\,\mathbf{k} \end{aligned}$$

- We also need to compute the vector field evaluated along the surface:

$$\mathbf{F}(g(\theta, \phi)) = \begin{pmatrix} \cos^2\theta\sin^2\phi \\ \cos\theta \\ -\sin\phi\sin\theta \end{pmatrix}$$

- Let's now compute the surface integral:

$$\begin{aligned} \int\int_S \mathbf{F} \cdot \mathbf{n} \, dA &= \int_0^{2\pi} \int_0^{\pi} \sin^2\theta\cos^2\theta\sin^2\phi\cos\phi + \sin^2\theta\sin\phi\cos\theta - \sin^2\theta\cos\theta\sin\phi \, d\theta \, d\phi \\ &= \int_0^{2\pi} \int_0^{\pi} \sin^2\theta\cos^2\theta\sin^2\phi\cos\phi \, d\theta \, d\phi \quad \text{Apply Fubini's theorem} \\ &= \left(\int_0^{\pi} \sin^2\theta\cos^2\theta \, d\theta \right) \left(\int_0^{2\pi} \sin^2\phi\cos\phi \, d\phi \right) \\ &= 0 \end{aligned}$$